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CONJUGATE FOURIER SERIES ON CERTAIN SOLENOIDS

BY

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ABSTRACT. We consider an arbitrary noncyclic subgroup of the additive group $\mathbf{Q}$ of rational numbers, denoted by $\mathbf{Q}_{\mathbf{a}}$, and its compact character group $\Sigma_{\mathbf{a}}$. For $1 < p < \infty$, an abstract form of Marcel Riesz's theorem on conjugate series is known. For f in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$, there is a function $\tilde{f}$ in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ whose Fourier transform $(\tilde{f})\hat{}(\alpha)$ at α in $\mathbf{Q}_{\mathbf{a}}$ is $-i \operatorname{sgn} \alpha \hat{f}(\alpha)$. We show in this paper how to construct $\tilde{f}$ explicitly as a pointwise limit almost everywhere on $\Sigma_{\mathbf{a}}$ of certain harmonic functions, as was done by Riesz for the circle group. Some extensions of this result are also presented.

1. Introduction

(1.1) *Notation*. This paper may be regarded as a sequel to [7], in which we established convergence and divergence theorems for Fourier series on a class of compact Abelian groups. The symbols $\mathbf{N}$, $\mathbf{Z}^+$, $\mathbf{Z}, \mathbf{Q}, \mathbf{R}$, and $\mathbf{C}$ denote the positive integers, nonnegative integers, integers, rational numbers, real numbers, and complex numbers, respectively. The symbol $\mathbf{T}$ denotes the circle group. We will parametrize $\mathbf{T}$ as $\{\exp(2\pi i t)\colon -\frac{1}{2} \leqslant t < \frac{1}{2}\}$. The symbols $[a, b]$ and $]a, b[$ denote the closed interval $\{t \in \mathbf{R}\colon a \leqslant t \leqslant b\}$ and the open interval $\{t \in \mathbf{R}\colon a < t < b\}$, respectively. Intervals $[a, b[$ and $]a, b]$ are defined similarly. For $z \in \mathbf{C}$, we write $\operatorname{sgn} z = z/|z|$ if $z \neq 0$ and $\operatorname{sgn} 0 = 0$. For $t \in \mathbf{R}$, $[t]$ denotes the greatest integer not exceeding t.

Letters $c, c', c_1, \ldots$ denote positive constants, which may vary from one occurrence to another.

We have to deal with integrals over four different measure spaces. To keep track, we will frequently write expressions like $\|f\|_{p,X}$, which means the $\mathfrak{L}_p$ norm of the function f over the measure space X. The symbol $\mathfrak{L}\log^+\mathfrak{L}(X)$ denotes the set of all measurable functions f on the measure space X such that $|f| \max(\log(|f|), 0)$ is in $\mathfrak{L}_1(X)$.

All notation not explained here is as in [8].

(1.2) *The groups* $\mathbf{Q}_{\mathbf{a}}$ *and* $\Sigma_{\mathbf{a}}$. We will study conjugate Fourier series on the character group of an arbitrary noncyclic subgroup of the additive group $\mathbf{Q}$. Up to isomorphisms, all such groups are described as follows. Let $\mathbf{a} = (a_0, a_1, a_2, \ldots, a_n, \ldots)$ be a fixed infinite sequence of integers all greater than 1. Let

$$(1) \qquad A_0 = 1, \quad A_1 = a_0, \quad A_2 = a_0 a_1, \ldots, \quad A_n = a_0 a_1 \cdots a_{n-1}, \ldots.$$

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Let $\mathbf{Q_a}$ be the set of all rational numbers l/A_k for $l \in \mathbf{Z}$ and $k \in \mathbf{Z}^+$. Plainly $\mathbf{Q_a}$ is a noncyclic additive subgroup of $\mathbf{Q}$. We will write $\mathbf{Q_a^+}$ for the semigroup of nonnegative numbers in $\mathbf{Q_a}$.

According to the Pontrjagin-van Kampen duality theory, the character group of $\mathbf{Q_a}$ is a compact Abelian group, which we denote by $\Sigma_{\mathbf{a}}$. The (continuous) character group of $\Sigma_{\mathbf{a}}$ is again $\mathbf{Q_a}$. We require a specific presentation of $\Sigma_{\mathbf{a}}$. First consider the group $\Delta_{\mathbf{a}}$ of **a**-adic integers. This group consists of all infinite sequences $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_n, \ldots)$ where each x_j belongs to the set $\{0, 1, \ldots, a_j - 1\}$. The sum $\mathbf{x} + \mathbf{y}$ is defined by adding coordinatewise and carrying quotients. See [**8**, Chapter II, §10, (10.2), p. 108] for details. Let $\mathbf{u}$ be the element $(1, 0, 0, \ldots, 0, \ldots)$ and $\mathbf{0}$ the element $(0, 0, 0, \ldots, 0, \ldots)$ in $\Delta_{\mathbf{a}}$.

We will later use the subgroups $\Lambda_0, \Lambda_1, \Lambda_2, \ldots$ of $\Delta_{\mathbf{a}}$, defined by $\Lambda_n = \{\mathbf{x} \in \Delta_{\mathbf{a}}: x_0 = x_1 = \cdots = x_{n-1} = 0\}$. Thus Λ_0 is $\Delta_{\mathbf{a}}$ itself, and $\Lambda_0 \supsetneqq \Lambda_1 \supsetneqq \Lambda_2 \supsetneqq \cdots$. Normalized Haar measure λ_n on Λ_n is the product of the measures $\sigma_j(A) = (1/a_j)\operatorname{card}(A)$ on the factors $\{0, 1, \ldots, a_j - 1\}$ of Λ_n.

We present the group $\Sigma_{\mathbf{a}}$ as the set $[-\frac{1}{2}, \frac{1}{2}[\times \Delta_{\mathbf{a}}$, with addition defined by

$$(2) \qquad (s, \mathbf{x}) \dot{+} (t, \mathbf{y}) = \left(s + t - \left[s + t + \tfrac{1}{2}\right], \mathbf{x} + \mathbf{y} + \left[s + t + \tfrac{1}{2}\right]\mathbf{u}\right).$$

The sets

$$(3) \qquad U_k(0, \mathbf{0}) = \{(t, \mathbf{x}) \in \Sigma_{\mathbf{a}}: |t| < 1/2k \text{ and } x_0 = x_1 = \cdots = x_{k-1} = 0\}$$

($k \in \mathbf{N}$) are a complete family of neighborhoods of the neutral element $(0, \mathbf{0})$ of $\Sigma_{\mathbf{a}}$. Normalized Haar measure μ on $\Sigma_{\mathbf{a}}$ is the product of Lebesgue measure λ on $[-\frac{1}{2}, \frac{1}{2}[$ and normalized Haar measure λ_0 on $\Delta_{\mathbf{a}} = \Lambda_0$.

Now consider any element $\alpha = l/A_j$ of $\mathbf{Q_a}$. We define χ_α as the complex-valued function on $\Sigma_{\mathbf{a}}$ such that

$$(4) \qquad \chi_\alpha(t, \mathbf{x}) = \exp\left[2\pi i \frac{l}{A_j}\left(t + \sum_{\nu=0}^{\infty} x_\nu A_\nu\right)\right],$$

where we agree that $\sum_{\nu=0}^{\infty} x_\nu A_\nu$ means $\sum_{\nu=0}^{j-1} x_\nu A_\nu$. It is easy to see that each χ_α is a continuous character of $\Sigma_{\mathbf{a}}$, that the characters χ_α separate the points of $\Sigma_{\mathbf{a}}$, that $\chi_{\alpha+\beta} = \chi_\alpha \chi_\beta$, and that χ_α is the function identically 1 if and only if $\alpha = 0$. Thus $\mathbf{Q_a}$ is the character group of $\Sigma_{\mathbf{a}}$ and $\Sigma_{\mathbf{a}}$ is the character group of $\mathbf{Q_a}$.

We will consider $\Delta_{\mathbf{a}}$ as being the subgroup $\{(0, \mathbf{x}): \mathbf{x} \in \Delta_{\mathbf{a}}\}$ of $\Sigma_{\mathbf{a}}$. The measures $\lambda_0, \lambda_1, \lambda_2, \ldots$ are thus singular probability measures in $\mathbf{M}(\Sigma_{\mathbf{a}})$. The characters χ_α defined in (4) are characters of $\Delta_{\mathbf{a}}$ as well as of $\Sigma_{\mathbf{a}}$ (simply compute $\chi_\alpha(0, \mathbf{x})$). It is clear that the χ_α comprise all of the continuous characters of $\Delta_{\mathbf{a}}$ and that the character group of $\Delta_{\mathbf{a}}$ is isomorphic with $\mathbf{Q_a}/\mathbf{Z}$. Thus we have a specific presentation of $\Sigma_{\mathbf{a}}$ and its characters, and are in a position to study in detail the behavior of Fourier and other trigonometric series on $\Sigma_{\mathbf{a}}$.

(1.3) *Conjugate Fourier series on* $\mathbf{T}$. In this paper, we extend to the groups $\Sigma_{\mathbf{a}}$ the principal facts about conjugate Fourier series on $\mathbf{T}$. We briefly recall these facts. Given f in $\mathfrak{L}_1(\mathbf{T})$ and $n \in \mathbf{Z}$, we write as usual

$$(1) \qquad \hat{f}(n) = \int_{-1/2}^{1/2} f(\exp(2\pi i t)) \exp(-2\pi i n t)\, dt.$$

THEOREM A (PRIVALOV [10, 11]). *Let f be a function in* $\mathfrak{L}_1(\mathbf{T})$. *Then*

$$\lim_{r\uparrow 1} \sum_{k=-\infty}^{\infty} -i \operatorname{sgn} k \hat{f}(k) r^{|k|} \exp(2\pi i k t) = \tilde{f}(\exp(2\pi i t)) \tag{2}$$

exists for almost all $t \in [-\frac{1}{2}, \frac{1}{2}[$. *The function* $\tilde{f}$ *is called* the conjugate function of f.

THEOREM B (MARCEL RIESZ [12]). *Let p be a number in the interval* $]1, \infty[$ *and let f be a function in* $\mathfrak{L}_p(\mathbf{T})$. *The function* $\tilde{f}$ *is in* $\mathfrak{L}_p(\mathbf{T})$, *and there is a constant* A_p *depending only on p such that*

$$\|\tilde{f}\|_p \leqslant A_p \|f\|_p. \tag{3}$$

(*As Titchmarsh* [15] *observed, we have* $A_p \sim cp$ *as* $p \uparrow \infty$ *and* $A_p \sim c/(p-1)$ *as* $p \downarrow 1$.)

THEOREM C (ZYGMUND [18]). *Suppose that f is in* $\mathfrak{L}\log^+\mathfrak{L}(\mathbf{T})$. *Then* $\tilde{f}$ *is in* $\mathfrak{L}_1(\mathbf{T})$ *and there are absolute constants c and c′ such that*

$$\|\tilde{f}\|_{1,\mathbf{T}} \leqslant c\int_{-1/2}^{1/2} |f(\exp(2\pi i t))| \log^+ |f(\exp(2\pi i t))| \, dt + c'. \tag{4}$$

(1.4) *Conjugate functions on compact Abelian groups with ordered duals.* Let X be a torsion-free infinite Abelian group, written additively and with the discrete topology. The group X contains a subset P such that $\mathsf{P} \cap (-\mathsf{P}) = \{0\}$, $\mathsf{P} \cup (-\mathsf{P}) = \mathsf{X}$, and $\mathsf{P} + \mathsf{P} = \mathsf{P}$. Defining $\chi \leqslant \psi$ and $\psi \geqslant \chi$ to mean that $\psi - \chi \in \mathsf{P}$, we see that $\leqslant$ is an order in X compatible with group addition. For details, see Rudin [13, pp. 193–195]. The ordering in X is never unique ($-\mathsf{P}$ will serve as well as P), and frequently X can be ordered in many quite different ways. Given X and P, we define the function sgn on X by

$$\operatorname{sgn} \chi = \begin{cases} 1 & \text{for } \chi \in \mathsf{P}\backslash\{0\}, \\ 0 & \text{for } \chi = 0, \\ -1 & \text{for } \chi \in -(\mathsf{P}\backslash\{0\}). \end{cases} \tag{1}$$

Let G be the (compact) character group of X. A complex-valued function f on G of the form

$$f = \sum_{\chi \in \mathsf{X}} a(\chi)\chi, \tag{2}$$

where a is a complex-valued function on X with finite support, is called *a trigonometric polynomial on G*, and the set of all trigonometric polynomials on G is denoted by $\mathfrak{T}(G)$.

There is an abstract version of Marcel Riesz's Theorem B.

THEOREM D (RUDIN [13, pp. 216–220] AND HELSON [6]). *Let f be a trigonometric polynomial on G, written in the form* (2). *The polynomial*

$$\tilde{f} = \sum_{\chi \in \mathsf{X}} -i \operatorname{sgn} \chi a(\chi)\chi \tag{3}$$

has the property that

$$\|\tilde{f}\|_{p,G} \leqslant A_p \|f\|_{p,G} \tag{4}$$

for all $p \in]1, \infty[$. *The constants* A_p *are the same as in Theorem* B *supra.*

THEOREM E. *Let* f *be a function in* $\mathfrak{L}_p(G)$ $(1 < p < \infty)$. *There is a function* $\tilde{f}$ *in* $\mathfrak{L}_p(G)$ *such that*

$$(\tilde{f})\hat{}(\chi) = -i \operatorname{sgn} \chi \hat{f}(\chi) \tag{5}$$

for all $\chi \in \mathsf{X}$. *The inequality* (4) *holds for* $\tilde{f}$ *and* f.

The function $\tilde{f}$ is called *the conjugate function of* f. Theorem E follows at once from Theorem D if one notes that $\mathfrak{T}(G)$ is an $\mathfrak{L}_p$-dense linear subspace of $\mathfrak{L}_p(G)$. The conjugate function $\tilde{f}$ is defined only as the limit of a certain sequence of trigonometric polynomials. Theorem E does not represent $\tilde{f}$ in any concrete way, as does Theorem A for the case $G = \mathbf{T}$ and $\mathsf{X} = \mathbf{Z}$.

(1.5) *The aim of this paper*. The group $\mathbf{Q}_{\mathbf{a}}$ admits exactly one order under which 1 is in P. We take this ordering for $\mathbf{Q}_{\mathbf{a}}$ and then have Theorems D and E. Our goal is to prove an analogue of Privalov's Theorem A for the group $\Sigma_{\mathbf{a}}$ and so to obtain the conjugate function of Theorem E explicitly almost everywhere on $\Sigma_{\mathbf{a}}$. As we will see, the existence of the analogue of (1.3)(2) is known only for functions in $\mathfrak{L}\log^+\mathfrak{L}(\Sigma_{\mathbf{a}})$.

2. The structure space of a certain commutative Banach algebra.

(2.1) *The classical case*. In their fundamental paper **[1]**, Arens and Singer pointed out the group-theoretic interpretation of the Poisson kernel for trigonometric series. The group $\mathbf{T}$ is the character group of $\mathbf{Z}$ and the closed disc $\mathbf{D} = \{z \in \mathbf{C}: |z| \leqslant 1\}$ is the semicharacter semigroup of the semigroup $\mathbf{Z}^+$. The Banach algebra $l_1(\mathbf{Z}^+)$ (under convolution) has $\mathbf{D}$ as its structure space and can be identified with the algebra of all functions $\sum_{n=0}^{\infty} a_n z^n$ on $\mathbf{D}$ with $\sum_{n=0}^{\infty} |a_n| < \infty$. The Poisson kernel can be thought of as the measure on $\mathbf{T}$ which, for all f in $l_1(\mathbf{Z}^+)$, can be convolved on $\mathbf{T}$ with the Gel'fand transform $\check{f}$ to give the value of $\check{f}$ at an arbitrary point of the structure space $\mathbf{D}$ of $l_1(\mathbf{Z}^+)$.

(2.2) *A certain structure space*. Let H be a commutative semigroup, written additively. The set of all complex-valued functions f on H such that

$$\|f\|_1 = \sum_{x \in H} |f(x)| < \infty$$

is denoted by $l_1(H)$. Plainly $l_1(H)$ is a complex Banach space with pointwise linear operations and the norm just described. With multiplication $f * g$ defined by

$$f * g(x) = \sum_{u,\, v:\, u+v=x} f(u)g(v), \tag{1}$$

$l_1(H)$ is a commutative Banach algebra.

A semicharacter of H is a bounded complex-valued function ζ on H that is not the zero function and has the property that $\zeta(x+y) = \zeta(x)\zeta(y)$ for all x, y in H. The multiplicative linear functionals on $l_1(H)$ are all defined by mappings τ_ζ for semicharacters ζ of H, where

$$\tau_\zeta f = \sum_{x \in H} f(x)\zeta(x). \tag{2}$$

These facts, by now familiar, are found (along with other matters) in Hewitt and Zuckerman [9]. Thus the structure space of $l_1(H)$ can be identified with the set of all semicharacters of H. The Gel'fand topology of this space is in general rather complicated, but is simple enough in the case we are concerned with.

To construct the Poisson kernel and the conjugate Poisson function (it is *not* a kernel, as we shall see) for the group $\Sigma_{\mathbf{a}}$, we need to find the structure space of the Banach algebra $l_1(\mathbf{Q}_{\mathbf{a}}^+)$. We omit the proofs of the next assertions, as they are straightforward. The semicharacters of $\mathbf{Q}_{\mathbf{a}}^+$ are the following functions:

(3) $$\psi_0, \text{ where } \psi_0(0) = 1 \quad \text{and} \quad \psi_0(\alpha) = 0 \text{ for } \alpha > 0;$$

(4) $$\alpha \mapsto \exp(-2\pi u\alpha)\chi_\alpha(t, \mathbf{x}) = \psi_{u,t,\mathbf{x}}(\alpha),$$

where u is a nonnegative real number and $(t, \mathbf{x})$ is a fixed element of $\Sigma_{\mathbf{a}}$. (That is, the function $\alpha \mapsto \chi_\alpha(t, \mathbf{x})$ is a character of $\mathbf{Q}_{\mathbf{a}}$ restricted to the subsemigroup $\mathbf{Q}_{\mathbf{a}}^+$.) The functions $\psi_{0,t,\mathbf{x}}$ obviously reproduce the group $\Sigma_{\mathbf{a}}$. Under our parametrization, the numbers u run through $[0, \infty[$, the numbers t through $[-\frac{1}{2}, \frac{1}{2}[$, and the sequences $\mathbf{x}$ through $\Delta_{\mathbf{a}}$, all independently.

Let $\Psi_{\mathbf{a}}$ denote the set consisting of the function (3) and all of the functions (4). As observed in the last paragraph but one, we can (and henceforth will) identify $\Psi_{\mathbf{a}}$ with the structure space of the commutative Banach algebra $l_1(\mathbf{Q}_{\mathbf{a}}^+)$.

For $f \in l_1(\mathbf{Q}_{\mathbf{a}}^+)$, we define its *Gel'fand transform* as the function $\check{f}$ on $\Psi_{\mathbf{a}}$ such that

(5) $$\check{f}(\psi_0) = f(0)$$

and

(6) $$\check{f}(\psi_{u,t,\mathbf{x}}) = \sum_{\alpha \in \mathbf{Q}_{\mathbf{a}}^+} f(\alpha)\psi_{u,t,\mathbf{x}}(\alpha).$$

The Gel'fand topology, which is the weakest topology on $\Psi_{\mathbf{a}}$ under which all of the functions $\check{f}$ are continuous, is the following. A generic neighborhood $V_n(\psi_0)$ of ψ_0 consists of ψ_0 and all $\psi_{u,t,\mathbf{x}}$ with $u \geqslant n$ $(n \in \mathbf{N})$. A generic neighborhood $V_n(\psi_{u,t,\mathbf{x}})$ of $\psi_{u,t,\mathbf{x}}$ consists of all $\psi_{v,s,\mathbf{y}}$ such that $v \in [0, \infty[$ and $|u - v| < 1/n$ and $(t, \mathbf{x}) \dot{-} (v, \mathbf{y}) \in U_n(0, \mathbf{0})$ in $\Sigma_{\mathbf{a}}$. Plainly $\Psi_{\mathbf{a}}$ is a compact Hausdorff space. It can be pictured as the Cartesian product $[0, \infty] \times \Sigma_{\mathbf{a}}$, where all of the points $\{\infty\} \times (t, \mathbf{x})$ are identified with each other.

The Šilov boundary of $\Psi_{\mathbf{a}}$ is the set of all semicharacters $\psi_{0,t,\mathbf{x}}$. Thus it can be identified with the compact group $\Sigma_{\mathbf{a}}$. This is proved, for example, in [**1**, Theorem 4.6].

3. The Poisson kernel for $l_1(\mathbf{Q}_{\mathbf{a}}^+)$.

(3.1) *Specifications for the Poisson kernel.* For typographical convenience, we will in the sequel write $\psi_{u,t,\mathbf{x}}$ as $(u, t, \mathbf{x})$ for $u > 0$ and $\psi_{0,t,\mathbf{x}}$ as $(t, \mathbf{x})$. We will continue to write ψ_0 as ψ_0. The *Poisson kernel* is a probability measure P_u on $\Sigma_{\mathbf{a}}$ for each $u \in [0, \infty]$ with the property that

(1) $$P_u * \check{f}(t, \mathbf{x}) = \check{f}(u, t, \mathbf{x})$$

for all $(u, t, \mathbf{x})$ $(0 \leqslant u < \infty)$ and

(2) $$P_\infty * \check{f}(t, \mathbf{x}) = \check{f}(\psi_0) = f(0)$$

for all $f \in l_1(\mathbf{Q}_{\mathbf{a}}^+)$.

(3.2) P_u *for* $0 < u < \infty$. The group $\Sigma_{\mathbf{a}}$ is the quotient group of $\mathbf{R} \times \Delta_{\mathbf{a}}$ by the subgroup $(1, -\mathbf{u})\mathbf{Z}$. Let φ be the canonical mapping of $\mathbf{R} \times \Delta_{\mathbf{a}}$ onto $\Sigma_{\mathbf{a}}$. The image under φ of the subgroup $\mathbf{R} \times \{\mathbf{0}\}$ is the subgroup S of $\Sigma_{\mathbf{a}}$ consisting of all elements $(t - [t + \frac{1}{2}], [t + \frac{1}{2}]\mathbf{u})$. Plainly φ is a continuous group isomorphism of $\mathbf{R} \times \{\mathbf{0}\}$ onto S. Of course φ is not a homeomorphism. Note also that $\mu(S) = 0$ and that S is dense in $\Sigma_{\mathbf{a}}$. We think of S as a sort of "spiral" lying densely in $\Sigma_{\mathbf{a}}$.

Given a measure ν in $\mathbf{M(R)}$ and a continuous complex-valued function f on $\Sigma_{\mathbf{a}}$, the integral

$$\int_{-\infty}^{\infty} f \circ \varphi(v)\, d\nu(v) = \nu_{\varphi}(f) \tag{1}$$

exists and defines a measure in $\mathbf{M}(\Sigma_{\mathbf{a}})$. Clearly all subsets of $\Sigma_{\mathbf{a}} \backslash S$ have $|\nu_{\varphi}|$ measure 0. For a positive real number u, let P_u be the measure in $\mathbf{M}(\Sigma_{\mathbf{a}})$ such that

$$\int_{\Sigma_{\mathbf{a}}} f(t, \mathbf{x})\, dP_u(t, \mathbf{x}) = \frac{1}{\pi} \int_{-\infty}^{\infty} f \circ \varphi(v) \frac{u}{u^2 + v^2}\, dv \tag{2}$$

for $f \in \mathfrak{C}(\Sigma_{\mathbf{a}})$. Clearly P_u is a probability measure singular with respect to Haar measure μ on $\Sigma_{\mathbf{a}}$.

To verify (3.1)(1) for all functions in $\check{l}_1(\mathbf{Q}_{\mathbf{a}}^+)$, it suffices to verify (3.1)(1) for functions of the form $\check{1}_{\{\alpha\}}$ $(\alpha \geqslant 0)$. We compute:

(3)

$$\begin{aligned}\int_{\Sigma_{\mathbf{a}}} \check{1}_{\{\alpha\}}(s, \mathbf{y})\, dP_u(s, \mathbf{y}) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \check{1}_{\{\alpha\}} \circ \varphi(v) \frac{u}{u^2 + v^2}\, dv \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \chi_{\alpha}(\varphi(v)) \frac{u}{u^2 + v^2}\, dv \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \exp\left(2\pi i \alpha \left(v - [v + \tfrac{1}{2}] + \sum_{\nu=0}^{\infty} x_{\nu} A_{\nu}\right)\right) \frac{u}{u^2 + v^2}\, dv,\end{aligned}$$

where $\mathbf{x} = (\mathbf{x}_{\nu})_{\nu=0}^{\infty}$ is the $\mathbf{a}$-adic expansion of the integer $[v + \frac{1}{2}]$. The last line of (3) is equal to

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \exp(2\pi i \alpha v) \frac{u}{u^2 + v^2}\, dv. \tag{4}$$

This can be proved by a calculation, or more easily by observing that the function $v \mapsto \chi_{\alpha}(\varphi(v))$ is a continuous character of $\mathbf{R}$ and so has the form $v \mapsto \exp(2\pi i s v)$ for some $s \in \mathbf{R}$. For $-\frac{1}{2} \leqslant v < \frac{1}{2}$, $\exp(2\pi i \alpha(v - [v + \frac{1}{2}] + \sum_{\nu=0}^{\infty} x_{\nu} A_{\nu}))$ is equal to $\exp(2\pi i \alpha v)$, which shows that the last line of (3) is equal to (4).

The integral (4) is of course equal to $\exp(-2\pi |\alpha| u) = \exp(-2\pi \alpha u)$. Thus we have

(5)

$$\begin{aligned}P_u * \check{1}_{\{\alpha\}}(t, \mathbf{x}) &= \int_{\Sigma_{\mathbf{a}}} \chi_{\alpha}((t, \mathbf{x}) \dot{-} (s, \mathbf{y}))\, dP_u(s, \mathbf{y}) \\ &= \chi_{\alpha}(t, \mathbf{x}) \overline{\int_{\Sigma_{\mathbf{a}}} \chi_{\alpha}(s, \mathbf{y})\, dP_u(s, \mathbf{y})} = \exp(-2\pi \alpha u) \chi_{\alpha}(t, \mathbf{x}) = \psi_{u, t, \mathbf{x}}(\alpha).\end{aligned}$$

Therefore (3.1)(1) holds for all f in $l_1(\mathbf{Q}_{\mathbf{a}}^+)$ and all u in $]0, \infty[$.

(3.3) *The measures P_0 and P_∞*. The Dirac measure $\delta_{(0,\mathbf{0})}$ is the multiplicative unit of $\mathbf{M}(\Sigma_{\mathbf{a}})$ and so for $u = 0$, (3.1)(1) holds with $P_0 = \delta_{(0,\mathbf{0})}$. Haar measure μ on $\Sigma_{\mathbf{a}}$ has the property that

$$\begin{aligned}\mu * \check{1}_{\{\alpha\}}(t,\mathbf{x}) &= \int_{\Sigma_{\mathbf{a}}} \chi_\alpha(t,\mathbf{x}) \dot{-} (s,\mathbf{y})) \, d\mu(s,\mathbf{y}) \\ &= \chi_\alpha(t,\mathbf{x}) \overline{\int_{\Sigma_{\mathbf{a}}} \chi_\alpha(s,\mathbf{y}) \, d\mu(s,\mathbf{y})} \\ &= \begin{cases} \chi_\alpha(t,\mathbf{x}) = 1 & \text{for } \alpha = 0, \\ 0 & \text{for } \alpha > 0. \end{cases}\end{aligned}$$

With $P_\infty = \mu$, the equality (3.1)(2) follows at once. Therefore P_∞ is Haar measure μ on $\Sigma_{\mathbf{a}}$.

(3.4) *The Gleason parts of $\Psi_{\mathbf{a}}$*. A theorem of Errett Bishop (see [**2**] or, for a detailed exposition, [**14**, p. 165, Theorem 16.6]) makes it easy to determine the Gleason parts of $\Psi_{\mathbf{a}}$. Namely, two points of $\Psi_{\mathbf{a}}$ lie in the same Gleason part if and only if their representing measures are mutually absolutely continuous. The representing measure of $(t, \mathbf{x})$ in $\Sigma_{\mathbf{a}}$ is plainly the Dirac measure $\delta_{(t,\mathbf{x})}$. The representing measure of $(u, t, \mathbf{x})$ with $0 < u < \infty$ is $P_u * \delta_{(t,\mathbf{x})}$. The representing measure of ψ_0 is Haar measure μ. Bishops's theorem shows that ψ_0 and the individual points of $\Sigma_{\mathbf{a}}$ are Gleason parts of $\Psi_{\mathbf{a}}$, each consisting of a single point.

The Gleason parts containing the points $(u, t, \mathbf{x})$ of $\Psi_{\mathbf{a}}$ with $0 < u < \infty$ are more interesting, and in fact are the sets on which we will carry out our detailed analysis. The measure $P_u * \delta_{(t,\mathbf{x})}$ is concentrated on the coset $S \dot{+} (t, \mathbf{x})$ of the subgroup S, and every Lebesgue measurable subset A of S with positive Lebesgue measure has the property that $P_u * \delta_{(t,\mathbf{x})}(A \dot{+} (t, \mathbf{x}))$ is positive. Therefore two representing measures $P_u * \delta_{(t,\mathbf{x})}$ and $P_v * \delta_{(s,\mathbf{y})}$ $(0 < u < \infty, 0 < v < \infty, (t, \mathbf{x}) \in \Sigma_{\mathbf{a}}, (s, \mathbf{y}) \in \Sigma_{\mathbf{a}})$ are mutually absolutely continuous if and only if $(t, \mathbf{x})$ and $(s, \mathbf{y})$ are in the same coset of S. That is, the nontrivial Gleason parts of $\Psi_{\mathbf{a}}$ are in one-to-one correspondence with the elements of the quotient group $\Sigma_{\mathbf{a}}/S$. It is easy to see that the cardinal number of this group is $\mathfrak{r}$. Its group-theoretic structure is complicated and does not concern us here. The quotient group topology is the trivial topology with exactly two open sets, and it too does not concern us. The point of interest to us is that the representing measures $P_u * \delta_{(t,\mathbf{x})}$ lie in a single Gleason part for fixed $(t, \mathbf{x})$ and all positive real numbers u.

4. The Poisson integral for $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$.

(4.1) *Remarks*. In the hands of classical analysts, the Poisson integral is defined *ab ovo* for all functions in $\mathfrak{L}_1(\mathbf{T})$ (indeed, for all measures in $\mathbf{M}(\mathbf{T})$). This, for example, is Zygmund's point of view [**19**, Chapter II, §§ 6–9 and *infra*]. We have followed Arens and Singer [**1**] and so have defined the Poisson integral as an integral over $\Sigma_{\mathbf{a}}$ only for functions in $\check{l}_1(\mathbf{Q}_{\mathbf{a}}^+)$, which is a very small subspace of $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$.

(4.2) *Construction*. To define the Poisson integral for all functions in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$, we use a theorem of abstract harmonic analysis. Given a locally compact group G, every measure ρ in $\mathbf{M}(G)$ can be convolved with every function f in $\mathfrak{L}_1(G)$ to produce a function $\rho * f$ in $\mathfrak{L}_1(G)$. This function can be written in the form

$$\rho * f(x) = \int_G f(y^{-1}x)\, d\rho(y), \tag{1}$$

the integral in (1) being a Lebesgue integral for almost all x in G (with respect to Haar measure on G). This is set forth in detail in Hewitt and Ross [**8**, Chapter V, Theorem (20.9), p. 290]. Applied to the group $\Sigma_{\mathbf{a}}$ and the measure P_u $(0 < u < \infty)$, (1) gives us

$$P_u * f(t,\mathbf{x}) = \int_{\Sigma_{\mathbf{a}}} f((t,\mathbf{x}) \dot{-} (s,\mathbf{y}))\, dP_u(s,\mathbf{y}) \tag{2}$$

for μ-almost all $(t,\mathbf{x})$ in $\Sigma_{\mathbf{a}}$. Standard theorems from integration theory and calculation in $\Sigma_{\mathbf{a}}$ lead from the integral (3.2)(2) to the equality

$$P_u * f(t,\mathbf{x}) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t - v - [t - v + \tfrac{1}{2}],[t - v + \tfrac{1}{2}]\mathbf{u} + \mathbf{x})\frac{u}{u^2 + v^2}\,dv. \tag{3}$$

A simple change of variable in (3) gives us

$$P_u * f(t,\mathbf{x}) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\varphi(v,\mathbf{x}))\frac{u}{u^2 + (t - v)^2}\,dv. \tag{4}$$

Suppose that the integral (4) exists as a Lebesgue integral for a certain choice of $\mathbf{x}$ in $\Delta_{\mathbf{a}}$, u_0 in $]0, \infty[$, and t_0 in $\mathbf{R}$. It is an elementary exercise to show that the integral $\int_{-\infty}^{\infty} f(\varphi(v,\mathbf{x}))\omega(v)\,dv$ exists as a Lebesgue integral for all bounded measurable functions ω on $\mathbf{R}$ for which $\omega(v) = \mathcal{O}(v^{-2})$ $(|v| \mapsto \infty)$. In particular, we have the following useful fact.

(4.3) THEOREM. *Suppose that the integral* (4.2)(4) *exists as a Lebesgue integral for some* $\mathbf{x}$ *in* $\Delta_{\mathbf{a}}$, *some* u_0 *in* $]0, \infty[$, *and some* t_0 *in* $\mathbf{R}$. *Then the integral* (4.2)(4) *exists for this* $\mathbf{x}$ *and for all* u *in* $]0, \infty[$ *and all* t *in* $\mathbf{R}$.

(4.4) REMARK. From (4.2) and (4.3), we see that for each $f \in \mathfrak{L}_1(\Sigma_{\mathbf{a}})$, there is a subset E_f of $\Delta_{\mathbf{a}}$ such that $\lambda_0(\Delta_{\mathbf{a}} \setminus E_f) = 0$ and such that the integral (4.2)(4) exists as a Lebesgue integral for all u in $]0, \infty[$, all t in $\mathbf{R}$, and all $\mathbf{x}$ in E_f.

(4.5) REMARK. A simple calculation shows that

$$P_u * f(\varphi(t + k,\mathbf{x} - k\mathbf{u})) = P_u * f(\varphi(t,\mathbf{x})) \tag{1}$$

for all $k \in \mathbf{Z}$ and all u, t, $\mathbf{x}$ for which (4.2)(4) exists.

(4.6) DEFINITION AND REMARKS. For all $\mathbf{x} \in E_f$, the integral (4.2)(4) is defined as *the Poisson integral of* f. It is defined for all $u > 0$ and all $t \in \mathbf{R}$ and so may be regarded as a complex-valued function defined in the upper half-plane $\mathbf{U} = \{t + iu: u > 0\}$ in the complex plane $\mathbf{C}$. We write this function on $\mathbf{U}$ as $P_u f(t,\mathbf{x})$ to distinguish it from $P_u * f(t,\mathbf{x})$, which is a function defined on $\Sigma_{\mathbf{a}}$.

(4.7) THEOREM. *The Poisson integral* $P_u f(t,\mathbf{x})$ *is a harmonic function of* $t + iu$ *in the upper half-plane* $\mathbf{U}$.

PROOF. An elementary theorem on differentiating integrals that depend upon a real parameter shows that

$$\Delta P_u f(t,\mathbf{x}) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\varphi(v,\mathbf{x}))\Delta\left(\frac{u}{u^2+(t-v)^2}\right) dv = 0. \quad \square$$

(4.8) THEOREM. *For all $\alpha \in \mathbf{Q_a}$, the Fourier transform of $P_u * f$ is given by*

(1) $$(P_u * f)\hat{}(\alpha) = \exp(-2\pi u|\alpha|)\hat{f}(\alpha).$$

PROOF. We have

$$\begin{aligned}\hat{P}_u(\alpha) &= \int_{\Sigma_\mathbf{a}} \overline{\chi_\alpha(t,\mathbf{x})}\, dP_u(t,\mathbf{x}) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\exp(-2\pi i\alpha v)\frac{u}{u^2+v^2}dv = \exp(-2\pi|\alpha|u).\end{aligned}$$

Now use the fact that the Fourier transform carries convolutions into pointwise products. $\square$

(4.9) THEOREM. *For every $\mathbf{x} \in E_f$, we have*

(1) $$\lim_{u\downarrow 0} P_u f(\varphi(t,\mathbf{x})) = f(\varphi(t,\mathbf{x}))$$

for almost all $t \in \mathbf{R}$.

PROOF. The function $v \mapsto f(\varphi(v,\mathbf{x}))/(1+v^2)$ is in $\mathfrak{L}_1(\mathbf{R})$, as Theorem (4.3) shows. The representation (4.2)(4) makes the validity of (1) a classical fact. See, for example, Titchmarsh [**16**, Chapter I, Theorem 1.17, pp. 30–31]. $\square$

5. The conjugate Poisson function on $\Psi_\mathbf{a}$.

(5.1) *Preliminaries*. The subgroups Λ_n of $\Delta_\mathbf{a}$, defined in (1.2), will now be regarded as subgroups of $\Sigma_\mathbf{a}$:

(1) $$\Lambda_n = \{(0,\mathbf{x}) \in \Sigma_\mathbf{a}: x_0 = x_1 = \cdots = x_{n-1} = 0\}.$$

The Haar measure λ_n will be regarded as a (singular) probability measure in $\mathbf{M}(\Sigma_\mathbf{a})$. It is easy to see that the Fourier-Stieltjes transform $\hat{\lambda}_n$ is the function $1_{(1/A_n)\mathbf{Z}}$ on $\mathbf{Q_a}$. The group $\Sigma_\mathbf{a}/\Lambda_n$ is topologically isomorphic with $\mathbf{T}$. The mapping

(2) $$(t,\mathbf{x}) \mapsto \exp\left(2\pi i\frac{1}{A_n}\left(t+\sum_{\nu=0}^{\infty} x_\nu A_\nu\right)\right) = \chi_{1/A_n}(t,\mathbf{x}) = \pi_n(t,\mathbf{x})$$

is a homomorphism of $\Sigma_\mathbf{a}$ onto $\mathbf{T}$ with kernel Λ_n.

A function f in $\mathfrak{L}_1(\Sigma_\mathbf{a})$ is constant on cosets of Λ_n if and only if $f = f * \lambda_n$. For every such function, there is a function g in $\mathfrak{L}_1(\mathbf{T})$ such that $f = g\circ\pi_n$, and we have

(3) $$\int_{\Sigma_\mathbf{a}} f\,d\mu = \int_{\Sigma_\mathbf{a}} g\circ\pi_n\,d\mu = \int_\mathbf{T} g\,d\lambda.$$

We shall have frequent recourse to the mapping (2) and the equalities (3).

(5.2) *The classical case*. For functions f in $\mathfrak{L}_1(\mathbf{T})$, the conjugate Poisson integral is defined by

$$(1)\qquad -i\sum_{k=-\infty}^{\infty}\operatorname{sgn} k r^{|k|}\hat{f}(k)\exp(2\pi ik\theta)$$
$$= \int_{-1/2}^{1/2} f(\exp(2\pi i(\theta - t)))\frac{2r\sin(2\pi t)}{1-2r\cos(2\pi t)+r^2}\,dt$$
$$= f * Q_r(\exp(2\pi i\theta)),$$

where as usual we write

$$(2)\qquad Q_r(\exp(2\pi it)) = 2r\sin(2\pi t)/\left(1-2r\cos(2\pi t)+r^2\right).$$

Note that

$$(3)\qquad (Q_r * f)\hat{}\,(l) = -i\operatorname{sgn} l r^{|l|}\hat{f}(l)$$

for all $l \in \mathbf{Z}$. We also have

$$(4)\qquad \|Q_r\|_{1,\mathbf{T}} = \frac{1}{\pi}\log\left(\frac{1+r^2}{1-r^2}\right).$$

Thus the conjugate Poisson integral for $\mathbf{T}$ is convolution with an absolutely continuous measure for each r such that $0 < r < 1$, although the norms of these measures go to ∞ as $r \uparrow 1$.

There is a fundamental difference between $\mathbf{T}$ and $\Sigma_{\mathbf{a}}$.

(5.3) THEOREM. *Let u be a positive real number. There is no measure, say σ_u, in $\mathbf{M}(\Sigma_{\mathbf{a}})$, such that*

$$(1)\qquad \hat{\sigma}_u(\alpha) = -i\operatorname{sgn}\alpha\exp(-2\pi u|\alpha|)$$

for all $\alpha \in \mathbf{Q}_{\mathbf{a}}$.

PROOF. Assume that there is a measure σ_u with property (1). As noted in (5.1), we have

$$(2)\qquad (\sigma_u * \lambda_n)\hat{}\,(\alpha) = 1_{(1/A_n)\mathbf{Z}}(\alpha)(-i\operatorname{sgn}\alpha)\exp(-2\pi u|\alpha|)$$

for all $n \in \mathbf{N}$. Observe also that

$$(3)\qquad \|\sigma_u * \lambda_n\| \leqslant \|\sigma_u\| < \infty.$$

We now regard $\mathbf{T}$ as $\pi_n(\Sigma_{\mathbf{a}})$, as in (5.1)(2). For a continuous complex-valued function f on $\mathbf{T}$, we have

$$(4)\qquad \int_{\Sigma_{\mathbf{a}}}(f\circ\pi_n)(\dot{-}\,(t,\mathbf{x}))\,d\sigma_u(t,\mathbf{x}) = (f\circ\pi_n) * \sigma_u(0,\mathbf{0})$$
$$= (f\circ\pi_n) * \lambda_n * \sigma_u(0,\mathbf{0}).$$

From (2) we see that $\lambda_n * \sigma_u$ is absolutely continuous and in fact is the function

$$(5)\qquad -i\sum_{k=-\infty}^{\infty}\operatorname{sgn} k\exp\left(-2\pi u\frac{|k|}{A_n}\right)\chi_{k/A_n}.$$

Thus the last line of (4) is equal to

$$(6)\qquad -i\sum_{k=-\infty}^{\infty}\operatorname{sgn}k\exp\left(-2\pi u\frac{|k|}{A_n}\right)\int_{\Sigma_{\mathbf{a}}}(f\circ\pi_n)(\dot{-}(t,\mathbf{x}))\chi_{k/A_n}(t,\mathbf{x})\,d\mu(t,\mathbf{x}).$$

The integrals appearing in (6) are equal to integrals over $\mathbf{T}$:

$$\int_{\Sigma_{\mathbf{a}}}(f\circ\pi_n)(\dot{-}\,(t,\mathbf{x}))\chi_{k/A_n}(t,\mathbf{x})\,d\mu(t,\mathbf{x})$$
$$=\int_{-1/2}^{1/2}f(\exp(-2\pi is))\exp(2\pi iks)\,ds=\hat{f}(k).$$

The sum (6) is thus equal to

$$(7)\qquad -i\sum_{k=-\infty}^{\infty}\operatorname{sgn}k\exp\left(2\pi u\frac{|k|}{A_n}\right)\hat{f}(k)$$
$$=\int_{-1/2}^{1/2}Q_{\exp(-2\pi u/A_n)}(\exp(2\pi is))f(\exp(-2\pi is))\,ds.$$

From (5.2)(4) and the definition of the norm of a measure, we see that the supremum of the absolute value of (7) over all continuous complex-valued functions f on $\mathbf{T}$ with $\|f\|_\infty\leqslant 1$ is

$$(8)\qquad \frac{1}{\pi}\log\left(\frac{1+\exp(-4\pi u/A_n)}{1-\exp(-4\pi u/A_n)}\right).$$

By (4), we see that $\|\sigma_u*\lambda_n\|$ is greater than or equal to the quantity (8). As $n\to\infty$, (8) goes to infinity, and this violates (3). Thus no measure σ_u with the stipulated property (1) exists. □

The preceding theorem shows that no conjugate Poisson kernel in the form of a family of measures on $\Sigma_{\mathbf{a}}$ can exist. Nonetheless, we can find a one-parameter *family of functions* that yield a concrete realization of the conjugate function provided by the abstract version of Marcel Riesz's Theorem E (1.4).

(5.4) Definitions and Remarks. Let f be a function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ and let E_f be as in (4.4). For $\mathbf{x}$ in E_f, t in $\mathbf{R}$, and $u>0$, let

$$(1)\qquad K_uf(t,\mathbf{x})=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\varphi(v,\mathbf{x}))\left[\frac{t-v}{u^2+(t-v)^2}+\frac{v}{1+v^2}\right]dv$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\varphi(v,\mathbf{x}))k(u,t,v)\,dv.$$

The function $k(u,t,v)$ is equal to

$$(2)\qquad \left((t-v)(1+tv)+u^2v\right)/\left(u^2+(t-v)^2\right)(1+v^2)$$

and so $K_uf(t,\mathbf{x})$ exists and is a Lebesgue integral for all positive u, all real t, and all $\mathbf{x}$ in E_f. We call $K_uf(t,\mathbf{x})$ *the conjugate Poisson function for* f. The term $v/(1+v^2)$ is a correction term used to secure convergence. It is similar to the correction term used in defining Hilbert transforms for bounded functions on $\mathbf{R}$: see for example Garnett [**4**, p. 109]. It is easy to see that $K_uf(t,\mathbf{x})$ is $\lambda\times\lambda\times\lambda_0$-measurable as a function of u, t, and $\mathbf{x}$.

(5.5) THEOREM. *For each* $\mathbf{x} \in E_f$, *the function* $t + iu \mapsto K_u f(t, \mathbf{x})$ *is harmonic in the upper half-plane* $\mathbf{U}$.

See the proof of Theorem (4.7).

(5.6) THEOREM. *For every* $\mathbf{x} \in E_f$, *the function* $t + iu \mapsto P_u f(t, \mathbf{x}) + iK_u(t, \mathbf{x})$ *is analytic in the upper half-plane* $\mathbf{U}$.

PROOF. As in the proof of Theorem (4.7), one sees immediately that this function satisfies the Cauchy-Riemann differential equations. □

(5.7) THEOREM. *For all* f *in* $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ *and all* $\mathbf{x} \in E_f$, *the limit*

$$\lim_{u \downarrow 0} K_u f(t, \mathbf{x}) = Kf(t, \mathbf{x}) \tag{1}$$

exists and is a complex number for almost all $t \in \mathbf{R}$.

PROOF. We use a device due to Littlewood, now standard in dealing with conjugate Fourier series and integrals. We may suppose that f is real and nonnegative. The function $P_u f(t, \mathbf{x})$ is nonnegative and the analytic function

$$\Phi(t + iu) = \exp(-(P_u f(t, \mathbf{x}) + iK_u f(t, \mathbf{x})))$$

is bounded in $\mathbf{U}$. By Fatou's theorem,

$$\lim_{u \downarrow 0} \Phi(t + iu) = \Gamma(t) \tag{2}$$

exists and is a complex number for almost all $t \in \mathbf{R}$. Since (4.9)(1) holds, the absolute value of $\Gamma(t)$, which is $\exp(-f(\varphi(t, \mathbf{x})))$, is positive for almost all $t \in \mathbf{R}$. A simple argument shows that $K_u f(t, x)$ must have a real-valued limit as $u \downarrow 0$ for almost all real t. □

(5.8) REMARKS. We make $Kf(t, x)$ into a function on $\Sigma_{\mathbf{a}}$ by restricting t to the interval $[-\frac{1}{2}, \frac{1}{2}[$, and then we may consider the Fourier transform $\widehat{Kf}(\alpha)$ for $\alpha \in \mathbf{Q}_{\mathbf{a}}$, should it happen that Kf is in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$. It is not the case that Kf is the conjugate function to f. That is, we do *not* have

$$\widehat{Kf}(\alpha) = -i \operatorname{sgn} \alpha \hat{f}(\alpha).$$

This is so because of the presence of the correction term $v/(1 + v^2)$ in the kernel $k(u, t, v)$ that appears in (5.4)(1). We will have to apply a corresponding correction term to Kf to get the conjugate function of f. We postpone this to §6. We turn now to some needed facts about $K_u f$ and Kf.

Throughout (5.9)–(5.12), p denotes a fixed but arbitrary number in the interval $]1, \infty[$. We first prove a technical lemma.

(5.9) LEMMA. *For* $u > 0$ *and* t, v *real, let*

$$\kappa(u, t, v) = \frac{(t - v)(1 + tv) + u^2 v}{(t - v)^2 + u^2}, \tag{1}$$

and let

$$N(u, v) = \left[\int_{-1/2}^{1/2} |\kappa(u, t, v)|^p \, dt\right]^{1/p}. \tag{2}$$

The quantities

$$S(u) = \sup\{N(u,v) : |v| \geq 3/2\} \tag{3}$$

are finite and are bounded for all u such that $0 < u \leq \frac{1}{4}$.

PROOF. Straightforward estimates show that

$$|\kappa(u,t,v)| \leq \frac{v^2 + (5/2 + 2u^2)|v| + 1}{(2|v|/3 + u)^2} \tag{4}$$

for all $u > 0$, $|t| \leq \frac{1}{2}$, $|v| \geq 3/2$. For $0 < u \leq \frac{1}{4}$, we have

$$|\tfrac{2}{3}|v| + u| \geq \tfrac{2}{3}|v| - u \geq \tfrac{2}{3}|v| - \tfrac{1}{6}|v| = \tfrac{1}{2}|v|.$$

The right side of (4) is thus less than or equal to $c + c'/|v| + c''/v^2$, which is plainly bounded for all v such that $|v| \geq 3/2$. Integrating the pth power from $-\frac{1}{2}$ to $\frac{1}{2}$, we find that $N(u,v)$ is bounded. □

(5.10) THEOREM. *Regard* $K_u f(t,x)$ *in* (5.4)(1) *as a function on* $\Sigma_{\mathbf{a}}$ *by restricting t to the interval* $[-\frac{1}{2}, \frac{1}{2}[$. *The mapping* $f \mapsto K_u f$ *is a bounded linear mapping of* $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ *into itself with the property that*

$$\|K_u f\|_p \leq C(p,u)\|f\|_p \tag{1}$$

for all f in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$, *where* $C(p,u)$ *is bounded for all u such that* $0 < u \leq \frac{1}{4}$.

PROOF. Let I denote the interval $[-\frac{3}{2}, \frac{3}{2}]$ and J the set $\mathbf{R} \setminus I$. We write

$$\begin{aligned} K_u f(t,\mathbf{x}) &= \int_J f(\varphi(v,\mathbf{x}))k(u,t,v)\,dv + \int_I f(\varphi(v,\mathbf{x}))\frac{t-v}{(t-v)^2+u^2}\,dv \\ &\quad + \int_I f(\varphi(v,\mathbf{x}))\frac{v}{1+v^2}\,dv \\ &= A_u(t,\mathbf{x}) + B_u(t,\mathbf{x}) + D(\mathbf{x}). \end{aligned} \tag{2}$$

We deal first with the function A_u. Let us apply the generalized Minkowski inequality (see, for example, [**19**, Chapter I, p. 19, (9.12)]). This inequality shows that

(3)
$$\begin{aligned} \|A_u\|_{p,\Sigma_{\mathbf{a}}} &= \left\{\int_{\Sigma_{\mathbf{a}}}\left|\int_J f(\varphi(v,\mathbf{x}))k(u,t,v)\,dv\right|^p d\mu(t,\mathbf{x})\right\}^{1/p} \\ &\leq \int_J\left\{\int_{\Sigma_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p|k(u,t,v)|^p\,d\mu(t,\mathbf{x})\right\}^{1/p} dv \\ &= \int_J\left\{\int_{\Sigma_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p|\kappa(u,t,v)|^p\,d\mu(t,\mathbf{x})\right\}^{1/p}\frac{dv}{1+v^2} \\ &= \int_J\left\{\int_{\Delta_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x})\right\}^{1/p}\left\{\int_{-1/2}^{1/2}|\kappa(u,t,v)|^p\,dt\right\}^{1/p}\frac{dv}{1+v^2}. \end{aligned}$$

Lemma (5.9) shows that the last line of (3) does not exceed

$$S(u)\int_J\left\{\int_{\Delta_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x})\right\}^{1/p}\frac{dv}{1+v^2}. \tag{4}$$

We now write

$$\begin{aligned}\int_J\left\{\int_{\Delta_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x})\right\}^{1/p}\frac{dv}{1+v^2} &= \sum_{|k|\geq 2}\int_{k-1/2}^{k+1/2}\left\{\int_{\Delta_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x})\right\}^{1/p}\frac{dv}{1+v^2}\\ &= \sum_{|k|\geq 2}\int_{-1/2}^{1/2}\left\{\int_{\Delta_{\mathbf{a}}}|f(\varphi(v+k,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x})\right\}^{1/p}\frac{dv}{1+(k+v)^2}.\end{aligned} \tag{5}$$

Since $\varphi(v+k,\mathbf{x})=\varphi(v,\mathbf{x}+k\mathbf{u})$ and since λ_0 is a translation-invariant measure on $\Delta_{\mathbf{a}}$, we see that

$$\int_{\Delta_{\mathbf{a}}}|f(\varphi(v+k,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x})=\int_{\Delta_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x}). \tag{6}$$

Hence the last line of (5) is less than or equal to

$$\sum_{|k|\geq 2}\frac{1}{1+(|k|-1/2)^2}\int_{-1/2}^{1/2}\left\{\int_{\Delta_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x})\right\}^{1/p}dv. \tag{7}$$

It is trivial that

$$\left\{\int_{\Delta_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x})\right\}^{1/p}\leq 1+\int_{\Delta_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x})$$

for all v in $[-\frac{1}{2},\frac{1}{2}[$. Hence (7) is less than or equal to

$$c\left[1+\int_{-1/2}^{1/2}\int_{\Delta_{\mathbf{a}}}|f(\varphi(v,\mathbf{x}))|^p\,d\lambda_0(\mathbf{x})\,dv\right]=c\left[1+\|f\|_p^p\right]. \tag{8}$$

Combining the estimates (7) through (3), we find that

$$\|A_u\|_p\leq 2S(u)c\|f\|_p, \tag{9}$$

as A_u is linear in f.

We next take up $B_u(t,\mathbf{x})$. For this function, we have

$$\|B_u\|_p^p=\int_{\Sigma_{\mathbf{a}}}\left|\int_{-3/2}^{3/2}f(\varphi(v,\mathbf{x}))\frac{t-v}{(t-v)^2+u^2}dv\right|^p d\mu(t,\mathbf{x}). \tag{10}$$

Write $f_{\mathbf{x}}$ for the function $v\mapsto f(\varphi(v,\mathbf{x}))$ for $v\in I$ and $v\mapsto 0$ for $v\in J$. The inner integral in (10) is equal to

$$\int_{-\infty}^{\infty}f_{\mathbf{x}}(v)\frac{t-v}{(t-v)^2+u^2}dv=\pi(f_{\mathbf{x}})\tilde{\ }(t+iu), \tag{11}$$

where the symbol "$\tilde{\ }$" denotes the classical conjugate Poisson integral, defined in $\mathbf{U}$. For $\mathbf{x} \in E_f$, (4.3) shows that the function

$$v \mapsto |f_{\mathbf{x}}(v)|^p/(1+v^2)$$

is in $\mathfrak{L}_1(\mathbf{R})$, and so $f_{\mathbf{x}}$ is in $\mathfrak{L}_p(\mathbf{R})$. Let $\mathbf{P}_u$ denote the classical Poisson kernel on $\mathbf{R}$. For an arbitrary function g in $\mathfrak{L}_p(\mathbf{R})$, we have

$$\tilde{g}(t+iu) = (\mathbf{P}_u * g)\tilde{\ }(t),$$

and so by M. Riesz's theorem, we get

$$\begin{aligned}(12)\qquad \int_{-\infty}^{\infty} |(f_{\mathbf{x}})\tilde{\ }(t+iu)|^p\,dt &= \int_{-\infty}^{\infty} |(\mathbf{P}_u * (f_{\mathbf{x}}))\tilde{\ }(t)|^p\,dt \\ &\leqslant A_p^p \|\mathbf{P}_u * f_{\mathbf{x}}\|_{p,\mathbf{R}}^p \leqslant A_p^p \|f_{\mathbf{x}}\|_{p,\mathbf{R}}^p.\end{aligned}$$

Taking (12) back into (10), we find that

$$\begin{aligned}(13)\qquad \|B_u\|_p^p &= \pi^p \int_{\Delta_{\mathbf{a}}} \int_{-1/2}^{1/2} |(f_{\mathbf{x}})\tilde{\ }(t+iu)|^p\,dt\,d\lambda_0(\mathbf{x}) \\ &\leqslant \pi^p \int_{\Delta_{\mathbf{a}}} \int_{-\infty}^{\infty} |(f_{\mathbf{x}})\tilde{\ }(t+iu)|^p\,dt\,d\lambda_0(\mathbf{x}) \\ &\leqslant \pi^p A_p^p \int_{\Delta_{\mathbf{a}}} \|f_{\mathbf{x}}\|_{p,\mathbf{R}}^p\,d\lambda_0(\mathbf{x}).\end{aligned}$$

The function $v \mapsto f_{\mathbf{x}}(v)$ on $\mathbf{R}$ vanishes outside of the interval $[-\frac{3}{2}, \frac{3}{2}]$, and as in (6) we see that

$$(14)\qquad \int_{\Delta_{\mathbf{a}}} \|f_{\mathbf{x}}\|_{p,\mathbf{R}}^p\,d\lambda_0(x) = 3\int_{\Delta_{\mathbf{a}}} \int_{-1/2}^{1/2} |f(\varphi(v,\mathbf{x}))|^p\,dv\,d\lambda_0(\mathbf{x}) = 3\|f\|_{p,\Sigma_{\mathbf{a}}}^p.$$

Combining (14) and (13), we find that

$$(15)\qquad \|B_u\|_p^p \leqslant c\|f\|_p^p.$$

Finally we estimate the p-norm of the function D. Noting that D depends only upon $\mathbf{x}$, we find that

$$\begin{aligned}(16)\qquad \|D\|_p^p &= \int_{\Delta_{\mathbf{a}}} \left| \int_{-3/2}^{3/2} f(\varphi(v,\mathbf{x})) \frac{v}{1+v^2}\,dv \right|^p d\lambda_0(\mathbf{x}) \\ &\leqslant 2^{-p} \int_{\Delta_{\mathbf{a}}} \left| \int_{-3/2}^{3/2} |f(\varphi(v,\mathbf{x}))|\,dv \right|^p d\lambda_0(\mathbf{x}) \\ &= 2^{-p} \int_{\Delta_{\mathbf{a}}} \left| 3\int_{-1/2}^{1/2} |f(\varphi(v,\mathbf{x}))|\,dv \right|^p d\lambda_0(\mathbf{x}) \\ &\leqslant \left(\frac{3}{2}\right)^p \int_{\Delta_{\mathbf{a}}} \int_{-1/2}^{1/2} |f(\varphi(v,\mathbf{x}))|^p\,dv\,d\lambda_0(\mathbf{x}) = \left(\frac{3}{2}\right)^p \|f\|_p^p.\end{aligned}$$

Take (16), (15), and (9) back to (2) to see that (1) holds, since $S(u)$ is bounded for u such that $0 < u \leqslant \frac{1}{4}$. □

A result somewhat like (5.10) holds for functions in $\mathfrak{L}\log^+\mathfrak{L}(\Sigma_{\mathbf{a}})$.

(5.11) THEOREM. *Let f be a function in $\mathfrak{L}\log^{+}\mathfrak{L}(\Sigma_{\mathbf{a}})$. Then the function $K_u f$ is in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ and we have*

$$(1)\qquad \|K_u f\|_1 \leqslant c + c\int_{\Sigma_{\mathbf{a}}} |f|\log^{+}(|f|)\,d\mu.$$

The constants c depend upon u and are bounded for all u such that $0 < u \leqslant \frac{1}{4}$.

PROOF. Examine the constant $C(p, u)$ in (5.10)(1), taking note of (5.10)(12) and the estimate for A_p given in Theorem B of (1.3). We see that

$$(2)\qquad C(p,u) = \mathcal{O}(1/(p-1))$$

for $1 < p < 2$, the estimate (2) being uniform in u for $0 < u \leqslant \frac{1}{4}$. We now cite a theorem of Yano [17], which appears in Zygmund [20, Chapter XII, (4.41), pp. 119–120]. This theorem shows that (2) and (5.10) imply (1). □

(5.12) THEOREM. *Let f be in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$. The function $Kf(t,\mathbf{x})$ defined in* (5.7)(1) *is also in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ and we have*

$$(1)\qquad \|Kf\|_p \leqslant cA_p\|f\|_p.$$

PROOF. Use the definition (5.7)(1), the inequalities (5.10)(1), and Fatou's lemma. □

(5.13) THEOREM. *Let f be in $\mathfrak{L}\log^{+}\mathfrak{L}(\Sigma_{\mathbf{a}})$. The function $Kf(t,\mathbf{x})$ of* (5.7)(1) *is in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ and we have*

$$(1)\qquad \|Kf\|_1 \leqslant c + c\int_{\Sigma_{\mathbf{a}}} |f|\log^{+}(|f|)\,d\mu.$$

PROOF. Apply (5.11)(1) and Fatou's lemma. □

6. The correction term.

(6.1) *Explanation*. We want to find a function $\mathbf{x} \mapsto C_f(\mathbf{x})$ on $\Delta_{\mathbf{a}}$ that can be subtracted from $K_u(t,\mathbf{x})$ to give a function whose Fourier transform behaves like the classical conjugate Poisson integral. That is, defining $\tilde{f}(t,\mathbf{x}) = K_u f(t,\mathbf{x}) - C_f(\mathbf{x})$, we want to have the result that

$$(1)\qquad (\tilde{f}_u)^{\wedge}(\alpha) = -i\,\mathrm{sgn}\,\alpha\exp(-2\pi u|\alpha|)\hat{f}(\alpha)$$

for all $\alpha \in \mathbf{Q}_{\mathbf{a}}$ and f in a reasonably large class of functions on $\Sigma_{\mathbf{a}}$.

(6.2) *Some computations*. Let f be a function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ and let n be a nonnegative integer. The function $f * \lambda_n$ can be written as $g \circ \pi_n$ for a function g in $\mathfrak{L}_1(\mathbf{T})$. As in the proof of (5.3), we write

$$(1)\qquad f * \lambda_n(\varphi(v,\mathbf{x})) = g(\pi_n(\varphi(v,\mathbf{x}))) = g\left(\exp\left(2\pi i\frac{1}{A_n}\left(v + \sum_{\nu=0}^{n-1} x_\nu A_\nu\right)\right)\right).$$

For positive integers N, we have

$$(2)\quad \frac{1}{\pi}\int_{-NA_n}^{NA_n}(f*\lambda_n)(\varphi(v,\mathbf{x}))\frac{v}{1+v^2}\,dv$$
$$=\frac{1}{\pi}\int_{-NA_n}^{NA_n} g\left(\exp\left(2\pi i\frac{1}{A_n}\left(v+\sum_{\nu=0}^{n-1}x_\nu A_\nu\right)\right)\right)\frac{v}{1+v^2}\,dv$$
$$=\frac{1}{\pi}\int_{0}^{A_n} g\left(\exp\left(2\pi i\frac{1}{A_n}\left(v+\sum_{\nu=0}^{n-1}x_\nu A_\nu\right)\right)\right)\left\{\sum_{k=-N}^{N-1}\frac{v+kA_n}{1+(v+kA_n)^2}\right\}dv.$$

The series appearing in $\{\cdots\}$ in the last line of (2) converges, as $N\to\infty$, to a function $\Psi(v)$, the convergence being uniform in the interval $[0, A_n]$. Accordingly we have

$$(3)\quad \lim_{N\to\infty}\frac{1}{\pi}\int_{-NA_n}^{NA_n}(f*\lambda_n)(\varphi(v,\mathbf{x}))\frac{v}{1+v^2}\,dv$$
$$=\frac{1}{\pi}\int_{0}^{A_n} g\left(\exp\left(2\pi i\frac{1}{A_n}\left(v+\sum_{\nu=0}^{n-1}x_\nu A_\nu\right)\right)\right)\Psi(v)\,dv.$$

To compute the function $\Psi(v)$, we recall the familiar equality

$$(4)\quad \int_{-\infty}^{\infty}\operatorname{sgn} x\exp(-|x|)\exp(-i(v+kA_n)x)\,dx=-2i\frac{v+kA_n}{1+(v+kA_n)^2}.$$

Defining

$$h(x)=\operatorname{sgn} x\exp(-|x|/A_n)\exp(-ivx/A_n),$$

we write the integral in (4) as

$$(5)\quad \frac{1}{A_n}\int_{-\infty}^{\infty}h(x)\exp(-ikx)\,dx.$$

Poisson's summation formula (see, for example, Zygmund [**19**, Chapter II, p. 68, (13.4)]) shows that

$$(6)\quad \Psi(v)=\sum_{k=-\infty}^{\infty}\frac{v+kA_n}{1+(v+kA_n)^2}$$
$$=\frac{\pi i}{A_n}\left(\sum_{k=-\infty}^{\infty}\operatorname{sgn} k\exp\left(\frac{-2\pi|k|}{A_n}\right)\exp\left(\frac{-2\pi ikv}{A_n}\right)\right).$$

The second line of (6) is equal to

$$(7)\quad \frac{2\pi}{A_n}\sum_{k=1}^{\infty}\left(\exp\left(\frac{-2\pi}{A_n}\right)\right)^k\sin\left(\frac{2\pi kv}{A_n}\right)=\frac{\pi}{A_n}Q_{\exp(-2\pi/A_n)}\left(\exp 2\pi i\frac{v}{A_n}\right).$$

Working back from (7) to (3), we find that

$$(8)\quad \lim_{N\to\infty}\frac{1}{\pi}\int_{-NA_n}^{NA_n} f*\lambda_n(\varphi(v,\mathbf{x}))\frac{v}{1+v^2}\,dv$$

$$=\frac{1}{A_n}\int_0^{A_n} g\left(\exp\left(2\pi i\frac{1}{A_n}\left(v+\sum_{\nu=0}^{n-1}x_\nu A_\nu\right)\right)\right)Q_{\exp(-2\pi/A_n)}\left(\exp 2\pi i\frac{v}{A_n}\right)dv.$$

The second integral in (8) is equal to

$$(9)\quad -\frac{1}{2\pi}\int_0^{2\pi} g\left(\exp\left(2\pi i\left(-\frac{w}{2\pi}+\frac{1}{A_n}\sum_{\nu=0}^{n-1}x_\nu A_\nu\right)\right)\right)Q_{\exp(-2\pi/A_n)}(\exp(iw))\,dw$$

$$=-Q_{\exp(-2\pi/A_n)}*g\left(\exp\left(2\pi i\left(\frac{1}{A_n}\sum_{\nu=0}^{n-1}x_\nu A_\nu\right)\right)\right).$$

Note that the convolution in (9) is on the group $\mathbf{T}$.

(6.3) DEFINITION AND REMARKS. For f in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ and n in $\mathbf{Z}^+$, we define $C_{f*\lambda_n}$ as the function on $\Delta_{\mathbf{a}}$ such that

$$(1)\qquad C_{f*\lambda_n}(\mathbf{x})=Q_{\exp(-2\pi/A_n)}*g\left(\exp\left(2\pi i\frac{1}{A_n}\left(\sum_{\nu=0}^{n-1}x_\nu A_\nu\right)\right)\right)$$

$$=\tilde{g}\left(\exp\left(\frac{-2\pi}{A_n}\right),\exp\left(2\pi i\frac{1}{A_n}\sum_{\nu=0}^{n-1}x_\nu A_\nu\right)\right).$$

The function g is defined in (6.2). For typographical convenience, we write $Q_r*f(\exp(2\pi i\theta))$ as $\tilde{f}(r,\exp(2\pi i\theta))$: see (5.2)(1). Observe that $C_{f*\lambda_n}$ is constant on cosets of Λ_n in $\Delta_{\mathbf{a}}$ and so is a continuous function on $\Delta_{\mathbf{a}}$ assuming only a finite number of values. Note too that the mapping $f\mapsto C_{f*\lambda_n}$ is linear in f. Our correction term C_f will be defined using the functions $C_{f*\lambda_n}$. We first estimate some $\mathfrak{L}_p(\Delta_{\mathbf{a}})$ norms.

(6.4) LEMMA. *Let p be a real number greater than 1 and let g be a function in $\mathfrak{L}_p(\mathbf{T})$. Let $\tilde{g}(r,\exp(2\pi i\theta))$ be as defined in (6.3) and let $\tilde{g}(1,\exp(2\pi i\theta))$ be $\lim_{r\uparrow 1}\tilde{g}(r,\exp(2\pi i\theta))$, the existence of which is guaranteed by Theorem* A *of* (1.3). *Let $P_r(\exp(2\pi i\theta))$ be the classical Poisson kernel on* $\mathbf{T}$:

$$(1)\qquad P_r(\exp(2\pi i\theta))=(1-r^2)/\left(1-2r\cos(2\pi\theta)+r^2\right).$$

We have

$$(2)\qquad |\tilde{g}(r,\exp(2\pi i\theta))|^p\leq\int_{-1/2}^{1/2}|\tilde{g}(1,\exp(2\pi it))|^pP_r(\exp(2\pi i(\theta-t)))\,dt.$$

PROOF. The function $\tilde{g}(r,\exp(2\pi i\theta))$ is harmonic in the open unit disc $\mathbf{D}$ and so the function $|\tilde{g}(r,\exp(2\pi i\theta))|^p$ is subharmonic: see, for example, Helms [**5**, p. 63, Example 7]. A well-known inequality for subharmonic functions on $\mathbf{D}$ states that

$$(3)\qquad |\tilde{g}(r,\exp(2\pi i\theta))|^p\leq\int_{-1/2}^{1/2}|\tilde{g}(a,\exp(2\pi it))|^pP_{r/a}(\exp(2\pi i(\theta-t)))\,dt$$

for all r such that $0 < r < a$, a being a fixed real number such that $0 < a < 1$. See, for example, Garnett [**4**, p. 37, Theorem 6.5]. As $a \uparrow 1$, $P_{r/a}(\exp(2\pi it))$ converges to $P_r(\exp(2\pi it))$, uniformly in t. We also have

$$(4) \qquad \lim_{a\uparrow 1}\int_{-1/2}^{1/2} |\tilde{g}(a, \exp(2\pi it)) - \tilde{g}(1, \exp(2\pi it))|^p\, dt = 0.$$

To see this, note that the maximal function $\sup_{0<a<1} |\tilde{g}(a, \exp(2\pi it))|$ is in $\mathfrak{L}_p(\mathbf{T})$ and apply dominated convergence. From (4), elementary estimates show that the limit as $a \uparrow 1$ of the right side of (3) exists and is equal to the right side of (2). □

(6.5) THEOREM. *Let p be a real number greater than 1 and let f be a function in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$. There is a positive constant c such that*

$$(1) \qquad \|C_{f*\lambda_n}\|_{p,\Delta_{\mathbf{a}}} \leqslant cA_p\|f\|_{p,\Sigma_{\mathbf{a}}}$$

for all positive integers n, where A_p is the constant in M. Riesz's Theorem B *in* (1.3).

PROOF. Observe first that as the sequences $(x_0, x_1, \ldots, x_{n-1})$ run through all values with $x_j \in \{0, 1, \ldots, a_j - 1\}$, the integers $\sum_{\nu=0}^{n-1} x_\nu A_\nu$ run through the integers $0, 1, \ldots, A_n - 1$, each value being assumed exactly once. Using (6.3)(1) and (6.4)(2), we thus have

$$(2) \qquad \begin{aligned} \int_{\Delta_{\mathbf{a}}} |C_{f*\lambda_n}(\mathbf{x})|^p\, d\lambda_0(\mathbf{x}) &= \frac{1}{A_n}\sum_{h=0}^{A_n-1}\left|\tilde{g}\left(\exp\left(\frac{-2\pi}{A_n}\right), \exp\left(2\pi i\frac{h}{A_n}\right)\right)\right|^p \\ &\leqslant \frac{1}{A_n}\sum_{h=0}^{A_n-1}\int_{-1/2}^{1/2} |\tilde{g}(1, \exp(2\pi it))|^p P_{\exp(-2\pi/A_n)}\left(\exp\left(2\pi i\frac{h}{A_n} - t\right)\right) dt \\ &= \int_{-1/2}^{1/2} |\tilde{g}(1, \exp(2\pi it))|^p \left\{\frac{1}{A_n}\sum_{h=0}^{A_{n-1}} P_{\exp(-2\pi/A_n)}\left(\exp\left(2\pi i\frac{h}{A_n} - t\right)\right)\right\} dt \\ &= \int_{-1/2}^{1/2} |\tilde{g}(1, \exp(2\pi it))|^p \Omega(n, t)\, dt; \end{aligned}$$

we have written the function appearing in $\{\cdots\}$ in the next to the last line of (2) as $\Omega(n, t)$.

We now estimate the nonnegative function $\Omega(n, t)$. For $r\exp(2\pi i\theta) = z$, we have

$$(3) \qquad P_r(\exp(2\pi i\theta)) = \mathrm{Re}\frac{1+z}{1-z} = 1 + 2\,\mathrm{Re}\sum_{k=1}^{\infty} z^k,$$

and so we see that

$$(4) \qquad \begin{aligned} \Omega(n, t) &= 1 + 2\,\mathrm{Re}\sum_{k=1}^{\infty}\frac{1}{A_n}\sum_{h=0}^{A_n-1}\exp\left(\frac{-2\pi k}{A_n}\right)\exp\left(2\pi ik\left(\frac{h}{A_n} - t\right)\right) \\ &= 1 + 2\sum_{k=1}^{\infty}\exp\left(\frac{-2\pi k}{A_n}\right)\cos(2\pi kt)\left\{\frac{1}{A_n}\sum_{h=0}^{A_n-1}\exp\left(2\pi i\frac{hk}{A_n}\right)\right\}. \end{aligned}$$

The sum $\{\cdots\}$ in (4) is zero if k/A_n is not an integer and is 1 if k/A_n is an integer. Thus we find that

$$(5)\qquad \Omega(n,t) = 1 + 2\sum_{l=1}^{\infty} \exp(-2\pi l)\cos(2\pi l A_n t) \leqslant \frac{1+\exp(-2\pi)}{1-\exp(-2\pi)}.$$

Applying (5) to (2) and using M. Riesz's Theorem B in (1.3), we obtain

$$(6)\qquad \| C_{f*\lambda_n} \|_{p,\Delta_{\mathbf{a}}}^p \leqslant c\int_{-1/2}^{1/2} |\tilde{g}(1,\exp(2\pi i t))|^p\,dt \leqslant cA_p^p \| g \|_{p,\mathbf{T}}^p$$
$$= cA_p^p \| f * \lambda_n \|_{p,\Sigma_{\mathbf{a}}}^p \leqslant cA_p^p \| f \|_{p,\Sigma_{\mathbf{a}}}^p.$$

The estimate (6) obviously gives (1). □

We have no analogue of Theorem (6.5) for $p = 1$. However, the following weaker estimate holds.

(6.6) THEOREM. *For each positive integer n, there is a positive number $A(n)$ such that the inequality*

$$(1)\qquad \| C_{f*\lambda_n} \|_{1,\Delta_{\mathbf{a}}} \leqslant A(n) \| f \|_{1,\Sigma_{\mathbf{a}}}$$

holds for all f in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$.

PROOF. Let

$$A(n) = \max\left\{ | Q_{\exp(-2\pi/A_n)}(\exp(2\pi i t)) | : -\tfrac{1}{2} \leqslant t < \tfrac{1}{2} \right\}.$$

Use (6.3)(1) to write

$$\| C_{f*\lambda_n} \|_{1,\Delta_{\mathbf{a}}} \leqslant A(n)\int_{-1/2}^{1/2} | g(\exp(2\pi i t)) |\,dt$$
$$= A(n)\int_{\Sigma_{\mathbf{a}}} | f * \lambda_n(t,\mathrm{x}) |\,d\mu(t,\mathrm{x}) \leqslant A(n) \| f \|_{1,\Sigma_{\mathbf{a}}}. \quad \square$$

We now show that the sequence of functions $(C_{f*\lambda_n})_{n=1}^{\infty}$ is a λ_0-martingale on $\Delta_{\mathbf{a}}$.

(6.7) DEFINITION AND REMARKS. For $n \in \mathbf{Z}^+$, let $\mathfrak{S}_n$ be the family of all cosets of Λ_n in $\Delta_{\mathbf{a}}$. Let $\mathfrak{F}_n$ be the (finite!) σ-algebra of subsets of $\Delta_{\mathbf{a}}$ generated by $\mathfrak{S}_n$. Plainly $\mathfrak{F}_n$ consists of 2^{A_n} sets, and is also the smallest σ-algebra of subsets of $\Delta_{\mathbf{a}}$ with respect to which the character χ_{1/A_n} is measurable.

(6.8) THEOREM. *Let f be a function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$. The sequence of functions $(C_{f*\lambda_n})_{n=1}^{\infty}$ is a λ_0-martingale on $\Delta_{\mathbf{a}}$ with respect to the increasing sequence of σ-algebras $(\mathfrak{F}_n)_{n=1}^{\infty}$.*

PROOF. It will suffice to prove that

$$(1)\qquad \int_{\Delta_{\mathbf{a}}} \chi_{k/A_{n-1}} C_{f*\lambda_{n-1}}\,d\lambda_0 = \int_{\Delta_{\mathbf{a}}} \chi_{k/A_{n-1}} C_{f*\lambda_n}\,d\lambda_0$$

for $n = 2,3,4,\ldots$ and $k = 0,1,2,\ldots,A_{n-1}-1$.

Suppose first that f is a character χ_{p/A_m} of $\Sigma_{\mathbf{a}}$. We take up first the case $m \geqslant 1$. We may suppose that p/A_m does not have the form q/A_{m-1} for any integer q. For every positive integer r, we have

$$(2)\qquad \chi_{p/A_m} * \lambda_r = \begin{cases} \chi_{p/A_m} & \text{for } r \geqslant m, \\ 0 & \text{for } r < m. \end{cases}$$

To see this, compute the Fourier-Stieltjes transform of the left side of (2). For $m \geqslant n+1$, (2) yields

$$\chi_{p/A_m} * \lambda_n = \chi_{p/A_m} * \lambda_{n-1} = 0,$$

so that both sides of (1) vanish. For $m \leqslant n-1$, (2) yields

$$\chi_{p/A_m} * \lambda_{n-1} = \chi_{p/A_m} * \lambda_n = \chi_{p/A_m},$$

and so the integrands on the two sides of (1) are identical. For $m = n$, we have

$$\chi_{p/A_m} * \lambda_{n-1} = 0,$$

and the left side of (1) vanishes. The right side of (1) is

$$(3) \qquad \int_{\Delta_{\mathbf{a}}} \chi_{k/A_{n-1}} C_{\chi_{p/A_n} * \lambda_n} \, d\lambda_0.$$

By (6.3)(1), the function $C_{\chi_{p/A_n} * \lambda_n}$ computed at $\mathbf{x}$ in $\Delta_{\mathbf{a}}$ has the value

$$(4) \qquad \int_{-1/2}^{1/2} \exp\left(2\pi i \frac{p}{A_n}\left(-t + \frac{1}{A_n}\sum_{\nu=0}^{n-1} x_\nu A_\nu\right)\right) Q_{\exp(-2\pi/A_n)}(\exp(2\pi i t))\, dt$$
$$= c\chi_{p/A_n}(\mathbf{x}).$$

Hence (3) is equal to

$$(5) \qquad c\int_{\Delta_{\mathbf{a}}} \chi_{k/A_{n-1}}(\mathbf{x})\chi_{p/A_n}(\mathbf{x})\, d\lambda_0(\mathbf{x}) = 0,$$

since characters are orthonormal under normalized Haar measure and the character χ_{p/A_n} cannot be the character $\overline{\chi_{k/A_{n-1}}}$. Therefore (1) holds if f has the form χ_{p/A_m} with $m \geqslant 1$. For characters χ_p of $\Sigma_{\mathbf{a}}$ with integral p, we have $\chi_p * \lambda_r = \chi_p$ for all $r \in \mathbf{N}$, and so (1) holds trivially for $f = \chi_p$ and all $n = 2, 3, \ldots$.

By linearity, (1) holds for all functions f in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ that are trigonometric polynomials. Let f be an arbitrary function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ and let $(p_l)_{l=1}^{\infty}$ be a sequence of trigonometric polynomials on $\Sigma_{\mathbf{a}}$ such that $\lim_{l\to\infty}\|f - p_l\|_{1,\Sigma_{\mathbf{a}}} = 0$. Theorem (6.6) shows that

$$\lim_{l\to\infty} \|C_{p_l * \lambda_n} - C_{f * \lambda_n}\|_{1,\Delta_{\mathbf{a}}} = 0.$$

Since (1) holds for all of the functions p_l, it thus also holds for f. □

(6.9) THEOREM. *Let p be a number greater than 1 and let f be a function in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$. The sequence of functions $C_{f*\lambda_n}$ on $\Delta_{\mathbf{a}}$ converges λ_0-almost everywhere on $\Delta_{\mathbf{a}}$ to a function C_f in $\mathfrak{L}_p(\Delta_{\mathbf{a}})$ for which the relations*

$$(1) \qquad \lim_{n\to\infty} \|C_{f*\lambda_n} - C_f\|_p = 0$$

and

$$(2) \qquad \|C_f\|_{p,\Delta_{\mathbf{a}}} \leqslant cA_p\|f\|_{p,\Sigma_{\mathbf{a}}}$$

obtain. Again, A_p is the constant in M. Riesz's Theorem B *in* (1.3).

PROOF. We use a classical theorem of martingale theory (Doob [**3**, Chapter VII, §4, p. 319, Theorem (4.1)]). Theorems (6.8) and (6.5) show that the hypotheses of part (i) of the cited theorem hold, and so the limit function C_f exists for λ_0-almost all points in $\Delta_{\mathbf{a}}$. Part (iii) of the cited theorem shows that (1) holds, and (2) is immediate from (1) and the estimate (6.5)(1). $\square$

(6.10) THEOREM. *Let f be a function in $\mathfrak{L}\log^+\mathfrak{L}(\Sigma_{\mathbf{a}})$. The martingale $(C_{f*\lambda_n})_{n=1}^{\infty}$ converges λ_0-almost everywhere in $\Delta_{\mathbf{a}}$ to a function C_f in $\mathfrak{L}_1(\Delta_{\mathbf{a}})$, for which we have*

$$(1)\qquad \|C_f\|_{1,\Delta_{\mathbf{a}}} \leq c + c\int_{\Sigma_{\mathbf{a}}} |f| \log^+(|f|)\, d\mu.$$

PROOF. For p near 1, we have $A_p \leq c/(p-1)$. Now use (6.5)(1) and Yano's theorem (Zygmund [**20**, Chapter XII, §4, p. 119, Theorem (4.41)]) to see that

$$(2)\qquad \int_{\Delta_{\mathbf{a}}} |C_{f*\lambda_n}|\, d\lambda \leq c + c\int_{\Sigma_{\mathbf{a}}} |f*\lambda_n| \log^+(|f*\lambda_n|)\, d\mu.$$

A simple argument, which we omit, shows that

$$(3)\qquad c + c\int_{\Sigma_{\mathbf{a}}} |f*\lambda_n| \log^+(|f*\lambda_n|)\, d\mu \leq c + c\int_{\Sigma_{\mathbf{a}}} |f| \log^+(|f|)\, d\mu.$$

From (2) and (3), we see that the $\mathfrak{L}_1(\Delta_{\mathbf{a}})$ norms of the functions $C_{f*\lambda_n}$ are bounded. Part (i) of the martingale theorem cited in the proof of Theorem (6.9) applies, giving both the existence of C_f and the inequality (1). $\square$

(6.11) DEFINITION AND REMARK. For f in $\mathfrak{L}\log^+\mathfrak{L}(\Sigma_{\mathbf{a}})$, let D_f be the subset of $\Delta_{\mathbf{a}}$ where $\lim_{n\to\infty} C_{f*\lambda_n}$ exists and is a complex number. Note that $\lambda_0(\Delta_{\mathbf{a}}\setminus D_f) = 0$.

7. Construction of the conjugate function.

(7.1) DEFINITION AND COMMENTS. Let f be a function in $\mathfrak{L}\log^+\mathfrak{L}(\Sigma_{\mathbf{a}})$. For $0 < u < \infty$, we define $\tilde{f}_u$ as the function on $\Sigma_{\mathbf{a}}$ whose value at $(t, \mathbf{x})$ is

$$(1)\qquad \tilde{f}_u(t,\mathbf{x}) = K_u f(t,\mathbf{x}) + C_f(\mathbf{x}),$$

and we define $\tilde{f}$ by

$$(2)\qquad \tilde{f}(t,\mathbf{x}) = Kf(t,\mathbf{x}) + C_f(\mathbf{x}).$$

We describe the sets where $\tilde{f}_u$ and $\tilde{f}$ are defined. Consider $\mathbf{x}$ in $E_f \cap D_f$, E_f being as in (4.4) and D_f as in (6.11). For each such $\mathbf{x}$, $K_u f(t,\mathbf{x})$ exists for all t in $\mathbf{R}$ and, in particular, for all t in $[-\frac{1}{2}, \frac{1}{2}[$. Thus $\tilde{f}_u$ is a μ-measurable function defined μ-almost everywhere on $\Sigma_{\mathbf{a}}$. By (5.7), $Kf(t,\mathbf{x})$ is defined for λ-almost all t in $\mathbf{R}$ and so for λ-almost all t in $[-\frac{1}{2}, \frac{1}{2}[$. Thus the functions $\tilde{f}_u(t, x)$ and $\tilde{f}(t, x)$ exist for μ-almost all $(t, \mathbf{x})$ in $\Sigma_{\mathbf{a}}$. We call $\tilde{f}$ *the conjugate function of f*.

(7.2) THEOREM. *The mappings $f \mapsto \tilde{f}_u$ and $f \mapsto \tilde{f}$ are linear mappings of $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ $(1 < p < \infty)$ and of $\mathfrak{L}\log^+\mathfrak{L}(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ with the following properties*:

$$(1)\qquad \|\tilde{f}_u\|_p \leq cA_p\|f\|_p,$$

$$(2)\qquad \|\tilde{f}\|_p \leq cA_p\|f\|_p,$$

$$(3)\qquad \|\tilde{f}_u\|_1 \leq c + c\int_{\Sigma_{\mathbf{a}}} |f| \log^+(|f|)\, d\mu,$$

$$\| \tilde{f} \|_1 \leqslant c + c\int_{\Sigma_{\mathbf{a}}} |f| \log^+(|f|)\, d\mu. \tag{4}$$

The constants c in (1) *and* (3) *depend upon u and are bounded for* $0 < u \leqslant \frac{1}{4}$.

PROOF. Linearity is obvious. For the inequalities (1)–(4), see (5.10)–(5.13), (6.9), and (6.11). □

We will now compute the Fourier transforms of $\tilde{f}_u$ and $\tilde{f}$.

(7.3) THEOREM. *Let α be any element of $\mathbf{Q}_{\mathbf{a}}$ and let u be a positive real number. Then we have*

$$(\chi_\alpha)_u^{\sim} = -i \operatorname{sgn} \alpha \exp(-2\pi |\alpha| u)\chi_\alpha. \tag{1}$$

PROOF. Write α as l/A_n, where n is a nonnegative integer and l is an integer. Since $\chi_\alpha * \lambda_m = \chi_\alpha$ for all $m \geqslant n$, the function C_{χ_α} is $C_{\chi_\alpha * \lambda_n}$. Using (7.1)(1), (5.4)(1), and (6.2), we have

$$\begin{aligned} (\chi_\alpha)_u^{\sim}(t,\mathbf{x}) &= K_u\chi_\alpha(t,\mathbf{x}) + C_{\chi_\alpha * \lambda_n}(\mathbf{x}) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \chi_\alpha(\varphi(v,\mathbf{x}))\left[\frac{t-v}{u^2+(t-v)^2} + \frac{v}{1+v^2}\right] dv \\ &\quad - \lim_{N\to\infty} \frac{1}{\pi}\int_{-NA_n}^{NA_n} \chi_\alpha(\varphi(v,\mathbf{x}))\frac{v}{1+v^2}\, dv \\ &= \lim_{N\to\infty} \frac{1}{\pi}\int_{-NA_n}^{NA_n} \chi_\alpha(\varphi(v,\mathbf{x}))\frac{t-v}{u^2+(t-v)^2}\, dv. \end{aligned} \tag{2}$$

It is easy to see that the last line of (2) is equal to the right side of (1). □

(7.4) THEOREM. *Let f be a function in $\mathfrak{L}\log^+\mathfrak{L}(\Sigma_{\mathbf{a}})$. For all $\beta \in \mathbf{Q}_{\mathbf{a}}$, we have*

$$(\tilde{f})^{\wedge}(\beta) = -i \operatorname{sgn} \beta \exp(-2\pi |\beta| u)\hat{f}(\beta) \tag{1}$$

and

$$(\tilde{f})^{\wedge}(\beta) = -i \operatorname{sgn} \beta \hat{f}(\beta). \tag{2}$$

PROOF. Both (1) and (2) hold if f is a trigonometric polynomial on $\Sigma_{\mathbf{a}}$: use linearity and Theorem (7.3). For f in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ for some $p > 1$, the density of trigonometric polynomials in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ proved (1) and (2). For f in $\mathfrak{L}\log^+\mathfrak{L}(\Sigma_{\mathbf{a}})$, one can use the *proof* of Yano's theorem that we cited in the proof of Theorem (6.10). From this proof it is easy to see that the validity of (1) and (2) for trigonometric polynomials implies its validity for all functions in $\mathfrak{L}\log^+\mathfrak{L}(\Sigma_{\mathbf{a}})$. □

(7.5) REMARK. The abstract conjugate function defined for f in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ by Theorem E of (1.4) is actually the function constructed in (7.1), since the two functions have the same Fourier transform.

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